A SIMPLE GEOMETRIC REPRESENTATIVE FOR μ OF A POINT

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Abstract. For $SU(2)$ (or $SO(3)$) Donaldson theory on a 4-manifold X, we construct a simple geometric representative for μ of a point. Let p be a generic point in X. Then the set $\{[A]|F_A^-(p)$ is reducible $\}$, with coefficient -1/4 and appropriate orientation, is our desired geometric representative.

1. Introduction

Let X be an oriented 4-manifold, let $G = SU(2)$ or $SO(3)$ and let B_k^* be the space of irreducible connections on P_k, the principal G bundle of instanton number k over X. Donaldson [D1, D2] defined a map $\mu : H_i(X, Q) \to H^{4-i}(B_k^*, Q)$. The images of the μ map are often described by geometric representatives: codimension $4-i$ varieties in B_k^* that are, roughly speaking, Poincare dual to classes in $H^{4-i}(B_k^*, Q)$. The form of these representatives depend on i.

If $Y \in H_3(X)$, then $\mu(Y)$ is related to the Chern-Simons functional of connections A evaluated on a 3-manifold representing Y. If $\Sigma \in H_2(X)$, then $\mu(\Sigma)$ is the Chern class of a line bundle over B^* defined using data on a surface representing Σ. If $\gamma \in H_1$, $\mu(\gamma)$ is related to the holonomy of connections around a loop representing γ. However, up to now there has not been any similar description of $\mu(x)$, where x is a generator of $H_0(X)$, in terms of data at a single point.

The purpose of this paper is to provide such a description. For any point $p \in X$, let $\nu_p = \{[A] \in B_k^* | F_A^-$ is reducible at $p\}$. Here $F_A^- = (F_A - *F_A)/2$ is the anti-self-dual part of the curvature F_A, and by "reducible at p" we mean that the components $F_{ij}^-(p)$ are all colinear as elements of the Lie algebra of G. The main theorem is

1991 *Mathematics Subject Classification*. 55R40, also 57R20.

This work is partially supported by an NSF Mathematical Sciences Postdoctoral Fellowship and Texas Advanced Research Project grant ARP-037

Theorem 1: *ν_p is a geometric representative of $-4\mu(x)$.*

The proof proceeds in stages. In section 2, we review some classical real algebraic geometry and construct a simple representative of p_1 of canonical $SO(3)$ bundles over Grassmannians of real oriented 3-planes. The construction is essentially due to Pontryagin [P] and Ehresmann [E], but their techniques seem to have been generally forgotten. In section 3, we extend this analysis to $BSO(3)$ and construct an explicit isomorphism between a space of connections on a neighborhood of the point p and $ESO(3)$. Pulling the representative of $p_1(ESO(3))$ back by this isomorphism gives ν_p, and fixes the orientation.

To be useful for Donaldson theory, ν_p must be transverse to the moduli spaces $\mathcal{M}_k$ and extend to the compactification of $\mathcal{M}_k$. This is discussed in section 4, where we also discuss a topological application of this representative.

2. Cohomology of Real Grassmannians

Let V_N be the space of real, rank 3, $3 \times N$ matrices. Equivalently, V_N is the Stieffel manifold of triples of linearly independent vectors in $\mathbb{R}^N$. Let V_N^0 be the triples of orthonormal vectors in $\mathbb{R}^N$. The group $SO(3)$ acts freely on both spaces by left multiplication. Let B_N be the quotient of V_N by $SO(3)$ and let G_N be the quotient of V_N^0 by $SO(3)$. G_N is the Grassmannian of oriented 3-planes in $\mathbb{R}^N$. We will denote by π both natural projections, from V_N to B_N and from V_N^0 to G_N. The Gramm-Schmidt process gives a natural bundle map from V_N to V_N^0, which we denote ρ. ρ itself defines trivial $\mathbb{R}^6$ bundles $V_N \to V_N^0$ and $B_N \to G_N$. Inclusion of V_N^0 in V_N defines a natural section. In short, we have the commutative diagram

$$\begin{array}{ccc} V_N & \xrightarrow{\rho} & V_N^0 \\ \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle \pi} \\ B_N & \xrightarrow{\rho} & G_N \end{array} \tag{1}$$

B_N and G_N have the same topology.

Theorem 2. *Let $\nu_N = \{m \in V_N |$ first 3 columns of m have rank $\leq 1\}$. Then $\pi(\nu_N)$ is Poincare dual to a generator of $H^4(B_N)$. By choosing orientations correctly, this generator may be taken to be the first Pontryagin class of the bundle $V_N \to B_N$.*

Proof: The proof is an application of some general computations of Pontryagin [P] and Ehresmann [E]. (Indeed, theorem 2 was almost certainly known to Pontryagin). Within the 9 dimensional space of real 3×3 matrices, the rank ≤ 1 matrices form a closed codimension-4 set. $\pi(\nu_N)$ is thus a closed codimension-4 submanifold of B_N, and so is dual to some (possibly zero) element of H^4. We construct a generator

of $H_4(B_N)$ and show it intersects $\pi(\nu_N)$ exactly once, establishing that $\pi(\nu_N)$ is a generator of H^4. The sign, relative to p_1, is determined separately.

We begin with a cell decomposition of G_N. Consider the set of $3 \times N$ matrices of the form

$$\begin{pmatrix} x_1 & x_2 & \dots & x_{i-1} & 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ y_1 & y_2 & \dots & y_{i-1} & 0 & y_{i+1} & \dots & y_{j-1} & 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ z_1 & z_2 & \dots & z_{i-1} & 0 & z_{i+1} & \dots & z_{j-1} & 0 & z_{j+1} & \dots & z_{k-1} & 1 & 0 & \dots & 0 \end{pmatrix}. \tag{2}$$

That is, a matrix with pivots $x_i = y_j = z_k = 1$, $i < j < k$, $y^i = z^i = z^j = 0$, and with no nonzero entries to the right of the pivots. Each oriented 3-plane corresponds to a unique matrix of this form, or to minus such a matrix. For fixed i, j, k we denote the set of matrices of this type as $e_+(i, j, k)$, and the set of negatives of these matrices as $e_-(i, j, k)$. The closures of the sets $e_\pm(i, j, k)$, called Schubert cycles, give a cellular decomposition of G_N.

The cell $e_+(i, j, k)$ has dimension $i + j + k - 6$. We give it the orientation $dx^1 \cdots dx^{i-1} dy^1 \cdots dy^{j-1} dz^1 \cdots dz^{k-1}$, where of course the variables y^i, z^i, z^j are skipped in this list. We orient $e_-(i, j, k)$ so the map $-1 : e_\pm(i, j, k) \to e_\mp(i, j, k)$ is orientation-preserving. The boundary map is then

$$\begin{aligned} \partial e_\pm(i, j, k) = &(-1)^i e_\pm(i-1, j, k) - e_\mp(i-1, j, k) \\ &+ (-1)^{i+j+1} e_\pm(i, j-1, k) + (-1)^i e_\mp(i, j-1, k) \\ &+ (-1)^{i+j+k+1} e_\pm(i, j, k-1) + (-1)^{i+j} e_\mp(i, j, k-1) \end{aligned} \tag{3}$$

This formula is of course independent of N.

$H_4(G_N)$ is then easily computed. It is $\mathbb{Z}$, and is generated by $S_N = e_+(1, 4, 5) + e_+(1, 3, 6) - e_+(1, 2, 7)$. The cycle $\rho(\pi(\nu_N))$ doesn't intersect $e_+(1, 3, 6)$ or $e_+(1, 2, 7)$, and hits $e_+(1, 4, 5)$ at exactly one point, namely

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & \dots & 0 \end{pmatrix}, \tag{4}$$

and the intersection is transverse. Thus $\rho(\pi(\nu_N))$ is a generator of $H^4(G_N)$. Pulling back we get that $\pi(\nu_N)$ is a generator of $H^4(B_N)$. All that remains is to fix the orientation such that $\pi(\nu_N)$ represents p_1.

To fix the orientation we consider the natural embedding $i : G_N \to G^{\mathbb{C}}_{3,N}$, where $G^{\mathbb{C}}_{3,N}$ is the Grassmannian of complex 3-planes in $\mathbb{C}^N$. The Pontyagin classes on

G_N are pullbacks of Chern classes on $G^{\mathbb{C}}_{3,N}$. In particular, $p_1 = -i^* c_2$ [MS]. We therefore have only to compute the intersection number in $G^{\mathbb{C}}_{3,N}$ of $i(S_N)$ with a cycle representing c_2. If W is a complex codimension-2 subspace of $\mathbb{C}^N$, then c_2 is represented by $Y \subset G^{\mathbb{C}}_{3,N}$, the set of 3-planes in $\mathbb{C}^N$ whose intersections with W have (complex) dimension 2 or greater [GH].

If $w_1, \ldots w_N$ are the natural coordinates on $\mathbb{C}^N$, we choose $W = \{w_1 + iw_4 = w_2 + iw_3 = 0\}$. A 3-plane spanned by the rows of

$$\begin{pmatrix} x_1 & x_2 & x_3 & x_4 & \ldots \\ y_1 & y_2 & y_3 & y_4 & \ldots \\ z_1 & z_2 & z_3 & z_4 & \ldots \end{pmatrix}, \tag{5}$$

is in Y if and only if the complex 3-vectors $(x_1 + ix_4,\ y_1 + iy_4,\ z_1 + iz_4)$ and $(x_2 + ix_3,\ y_2 + iy_3,\ z_2 + iz_3)$ are (complex) colinear. This is never the case in the closures of $e_+(1,3,6)$ or $e_+(1,2,7)$.

Matrices in $e_+(1,4,5)$ take the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 \\ 0 & y_2 & y_3 & 1 & 0 & 0 & \ldots & 0 \\ 0 & z_2 & z_3 & 0 & 1 & 0 & \ldots & 0 \end{pmatrix}, \tag{6}$$

Y intersects $e_+(1,4,5)$ at the single point $y_2 = y_3 = z_2 = z_3 = 0$, and the intersection number is easily computed to be $+1$.

Thus for a cycle on G_N (or B_N) to represent p_1, it must be oriented to intersect S_N (or its image under the natural section) negatively. This completes the proof of theorem 2.

3. Evaluation of $\mu(p)$.

The finite-dimensional results of section 2 cannot be directly applied to gauge theory. We need to extend them to appropriate infinite-dimensional spaces. Let H be an infinite-dimensional Banach space. Pick an infinite sequence of linearly independent vectors in H. Then there are natural inclusions

$$\mathbb{R}^N \overset{i}{\hookrightarrow} \mathbb{R}^{N+1} \overset{i}{\hookrightarrow} \cdots \overset{i}{\hookrightarrow} \mathbb{R}^\infty \overset{i}{\hookrightarrow} H, \tag{7}$$

where $\mathbb{R}^\infty$ is the direct limit of the spaces $\mathbb{R}^N$. This induces a sequence of inclusions

$$V_N \overset{i}{\hookrightarrow} V_{N+1} \overset{i}{\hookrightarrow} \cdots \overset{i}{\hookrightarrow} V_\infty \overset{i}{\hookrightarrow} V_H \tag{8}$$

and corresponding inclusions for V^0, B and G. For N large, these inclusions induce isomorphisms in H_4 (see e.g. [MS]), sending S_N to S_{N+1} to ... to S_∞ to S_H. $\pi(\nu_\infty)$ is closed and intersects S_∞ once, and $\pi(\nu_H)$ is closed and intersects S_H once. By the same argument as before, we have

Theorem 3 $\pi(\nu_\infty)$, *oriented so as to intersect* S_∞ *negatively, represents* p_1 *of the bundle* $V_\infty \to B_\infty$, *and* $\pi(\nu_H)$, *oriented to intersect* S_H *negatively, represents* p_1 *of* $V_H \to B_H$.

An equivalent description of p_1 is as follows. Let W be a codimension-3 subspace in H. Let Y_W be the set of 3-frames whose span, intersected with W, is at least 2-dimensional. When $W = \{x_1 = x_2 = x_3 = 0\}$, $Y_W = \nu_H$. But, since G_{H^*} is connected, the choice of W cannot affect the topology of Y_W. Thus Y_W, oriented to intersect S_H negatively, represents p_1 for any choice of W.

We are now able to construct μ of a point. Let p be a point on the manifold X, let D be a geodesic ball around p, let $\mathcal{A}_D$ be the $SU(2)$ (or $SO(3)$) connections on D within the Sobolev space L^q_k (the choice of q and k is not important), let $\mathcal{G}^0$ be the gauge transformations in L^q_{k+1} that leave the fiber at p fixed, and let $\mathcal{G}$ be all gauge transformations in L^q_{k+1}. Define $\mu_D(p)$ to be $-\frac{1}{4}p_1$ of the $SO(3)$ bundle $\mathcal{A}_D/\mathcal{G}^0 \to \mathcal{A}_D/\mathcal{G}$. $\mu(p)$ is the pullback of $\mu_D(p)$ to $B(X)$ via the map that restricts connections on a bundle over X to a bundle over D.

The space $\mathcal{A}_D/\mathcal{G}^0$ is isomorphic to the set of connections in radial gauge with respect to the point p. In such a gauge the connection form A vanishes in the radial direction but is otherwise unconstrained. In particular, $A(p) = 0$, so the curvature at p, $F_A(p) = dA(p) + A(p) \wedge A(p) = dA(p)$, is a linear function of A.

Let H be the space of (scalar valued) 1-forms with no radial component. A connection in radial gauge is defined by a triple of elements of H, one for each direction in the Lie Algebra. Deleting the infinite-codimension set for which these elements are linearly dependent we get V_H. Thus $\mu_D(p)$ is -1/4 times p_1 of $V_H \to B_H$, which we have already computed. Let $W = \{\alpha \in H | d^-\alpha(p) = 0\}$. Thus Y_W is the set of connections over D, in radial gauge, for which the three components of $F_A^-(p)$ span a 1 (or 0) dimensional subspace of the Lie algebra. In other words, for which $F_A^-(p)$ is reducible. Pulling $\mu_D(p)$ back by the restriction map we get the connections on X for which $F_A^-(p)$ is reducible, i.e. ν_p. This completes the proof of theorem 1.

4. Transversality and Extension to the Boundary.

We have shown that for any point p in our manifold, the cycle ν_p is Poincare dual to p_1 of the base point fibration, as a class in $\mathcal{B}^*(X)$. However, to do Donaldson

theory we need more than this. We need for ν_p to intersect the moduli space $\mathcal{M}_k$ transversely and to extend in a well-behaved way to the compactification of moduli space. Had we chosen ν_p to depend on F_A^+ rather than F_A^-, it would still have been dual to p_1, but would have been useless as a geometric representative of $-4\mu(p)$, insofar as F_A^+ is identically zero on $\mathcal{M}_k$.

Even with our definition of ν_p, it is unrealistic to expect ν_p to intersect $\mathcal{M}_k$ transversely for all points p. For example, if $\mathcal{M}_k$ has dimension $d < 4$, then transversality would imply that $\nu_p \cap \mathcal{M}_k = \emptyset$. However, there is a $d+4$ dimensional set of pairs (A, p) for which $F_A^-(p)$ might be reducible. Since reducibility is a codimension-4 condition, we should expect reducibility at a d-dimensional set of pairs. Thus for p in a d-dimensional subset of X, ν_p would not intersect $\mathcal{M}_k$ transversely. There is no reason to suppose that this d-dimensional set is empty.

A more reasonable outcome is the following:

Conjecture: *For $k > 0$, generic points p and generic metrics, ν_p intersects $\mathcal{M}_k$, and $\mathcal{M}_k$ cut down by standard Donaldson varieties, transversely.*

Should this conjecture prove true, then non-transverse intersection points (for generic metrics) can always be resolved by moving p. If the conjecture is not true, then we will require more subtle means of perturbing ν_p or $\mathcal{M}_k$. The utility of the representative ν_p will depend on how easy or difficult this turns out to be.

The case $k = 0$ yields special difficulties. First, $\mathcal{M}_0$ contains the trivial connection (and other reducible connections if $H_1(X) \neq 0$), and so is not contained in B_0^*. This complication is independent of the choice of representative of $\mu(x)$ and is not discussed here.

The second complication is that every flat connection is in ν_p, so that ν_p cannot possibly intersect $\mathcal{M}_0$ transversely. To resolve this we must perturb $\mathcal{M}_0$. If $\pi_1 = 0$, so that $\mathcal{M}_0$ is just the trivial connection, this is easy. We just add a small connection that is zero outside a small neighborhood of p. One can always find a connection for which $F_A^-(p)$ will be irreducible, so ν_p will miss the perturbed $\mathcal{M}_0$ entirely. If $\pi_1 \neq 0$ and $\mathcal{M}_0$ contains a higher-dimensional representation variety, it may happen that one cannot lift $\mathcal{M}_0$ entirely off ν_p. In that case we must interpret "$\mathcal{M}_0 \cap \nu_p$" as the intersection points that remain after a fixed (but generic) infinitesimal perturbation of $\mathcal{M}_0$.

Next we consider the extension of ν_p to the compactification of $\mathcal{M}_k$. The boundary of $\mathcal{M}_k$ consists of strata where m instantons have pinched off, leaving a solution of charge $k - m$ behind. These take the form $\mathcal{M}_{k-m} \times S^m(X)$, where $m > 0$. These boundary strata have lower dimension than $\mathcal{M}_k$, so they *should* not contribute to Donaldson invariants. To ensure that they do not contribute, we must show that

ν_p remains a codimension-4 set on the boundary. Similar theorems are known for other, more well-known, representatives of Donaldson cycles (e.g. Proposition 5.3.2. of [DK]).

Theorem 4: *The intersection of the closure of ν_p with $\partial\mathcal{M}_k$ is contained in the union of the following three sets:*

1. $(\nu_p \cap \mathcal{M}_{k-m}) \times S^m(X)$, *with $m < k$.*
2. $\mathcal{M}_{k-m} \times \{p\} \times S^{m-1}(X)$.
3. $\mathcal{M}_0 \times S^k(X)$.

Proof: Consider a sequence of connections $[A_i] \in \mathcal{M}_k \cap \nu_p$ converging to $[A'] \times \{x_1, \ldots, x_m\}$, where $[A'] \in \mathcal{M}_{k-m}$. If $p \notin \{x_i\}$, then $F^-_{A_i}(p)$ converges, after suitable gauge transformations, to $F^-_{A'}(p)$. Since the set of rank ≤ 1 matrices is closed and invariant under left multiplication by $SO(3)$ (i.e. gauge transformations), $F^-_{A'}$ has rank at most 1, and we have case 1 or 3, depending on whether $m < k$ or $m = k$. If $p \in \{x_i\}$ we have case 2. QED.

For ν_p, cases 1 and 2 will give us no trouble, as the codimension of ν_p in each stratum remains 4. In case 3 we must perturb $\mathcal{M}_0$ to make sense of $\mathcal{M}_0 \cap \nu_p$, after which the resulting set still has codimension 4.

We close with a sketch of a topological application of this geometric representative. The details will appear elsewhere [GS].

By definition, 4-manifolds of "simple type" have Donaldson invariants that satisfy a certain recursion relation. This relation roughly says that, given two points p and q, $\mathcal{M}_k \cap \nu_p \cap \nu_q$ has the same top homology as $64\mathcal{M}_{k-1}$. For p and q close, a possible mechanism for this behavior would be that, given any point A in $\mathcal{M}_{k-1}$, there should be 64 ways to glue in a concentrated instanton near p and q to make the curvature at p and q reducible. The curvature at p and q is easily expressible in terms of the curvature of A and the parameters of the gluing map, allowing us to test this hypothesis. It is false. For generic A there are only 6 ways, not 64, to make $F^-_A(p)$ and $F^-_A(q)$ reducible. Thus the interior of $\mathcal{M}_k \cap \nu_p \cap \nu_q$ contains 58 (or 70) copies of $\mathcal{M}_{k-1}$, up to homology. This is surprising, insofar as the only known embeddings of $\mathcal{M}_{k-1}$ in $\mathcal{M}_k$ involve Taubes patching, and so have image near the boundary of $\mathcal{M}_k$.

I thank Stefan Cordes, Dan Freed, David Groisser, Takashi Kimura, Rob Kusner, Tom Parker, Jan Segert, Cliff Taubes and Karen Uhlenbeck for extremely helpful discussions. Part of this work was done at the 1994 Park City/IAS Mathematics Institute. This work is partially supported by an NSF Mathematical Sciences Postdoctoral Fellowship and by Texas Advanced Research Program grant ARP-037.

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